ABSTRACT

DEPARTMENT OF PHYSICS

BURNETTE, JR., JAMES E. B.S., MOREHOUSE COLLEGE, 1993

THE NUMBER OF LIMIT-CYCLES FOR THE GENERALIZED, MIXED RAYLEIGH-LIENARD OSCILLATOR EQUATION

Advisor: Professor Ronald E. Mickens

Thesis dated July, 1995

We investigate the issue of how many limit-cycles can exist for the generalized, mixed Rayleigh-Liénard differential equation. This equation is an ordinary, second-order, nonlinear differential equation and arises in the modeling of a variety of systems in the natural and engineering sciences. A method of averaging is used to determine the maximum number of limit-cycles that can exist for this equation. We compare our results to those obtained by two other analytical methods. We also discuss a new mathematical phenomena that arises in the use of averaging methods, namely, the existence of "spurious" limit-cycles.

THE NUMBER OF LIMIT-CYCLES FOR THE GENERALIZED, MIXED RAYLEIGH-LIENARD OSCILLATOR EQUATION

A THESIS

SUBMITTED TO THE FACULTY OF CLARK ATLANTA UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF MASTER OF SCIENCE

BY

JAMES E. BURNETTE, JR.

DEPARTMENT OF PHYSICS

ATLANTA, GEORGIA

JULY 1995

ACKNOWLEDGEMENTS

I would especially like to thank Professor Ronald E. Mickens for his encouragement, support, inspiration, and understanding. I also thank Professor Lonzy Lewis for his continued support during my matriculation at Clark Atlanta University. Additionally, thanks goes to my best friend, Carmella Allen; my son, Jeremy; my parents and all of those who have looked out for me along the way.

The research of this thesis was supported in part by grants from NASA and NIH (MBRS).

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

1.1 Statement of the Problem

One dimensional oscillatory systems can be modeled by nonlinear, ordinary differential equations that take the form[1,2]

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right), \tag{1.1}$$

where F is generally a rational function of its arguments. This equation can be rewritten as a set of coupled first-order equations as follows

$$\frac{dx}{dt} = y, \tag{1.2a}$$

$$\frac{dy}{dt} = F(x, y). \tag{1.2b}$$

The (x, y) plane is called the phase-plane[2] and the phase trajectories, $y = y(x)$, are solutions to the differential equation[1,2]

$$\frac{dy}{dx} = \frac{F(x, y)}{y}. \tag{1.3}$$

A simple closed curve in the phase-plane that has nearby nonclosed trajectories spiraling toward it, either as $t \to \infty$ or as $t \to -\infty$, is called a limit-cycle.[1,2] A stable limit-cycle has all trajectories that start sufficiently close to it, both inside and outside, spiral toward it as $t \to \infty$. If trajectories on both sides spiral away as $t \to \infty$, the limit-cycle is unstable.

The generalized mixed, Rayleigh-Liénard equation is[3,4,5]

$$\ddot{x} + A_3 x + B_3 x^3 = (D_3 + E_{31} x^2 + E_{32} \dot{x}^2 + F_3 x^4)\dot{x}, \tag{1.4}$$

where the "dot" represents the time derivative, and $(A_3, B_3, D_3, E_{31}, E_{32}, F_3)$ are constants. Our research problem was to use the averaging method of Krylov-Bogoliubov-Mitropolsky (KBM)[6,7] to determine the maximum number of limit-cycles for Eq. (1.4).

1.2 Summary of Previous Results

The work of this thesis was inspired by the recent appearance of two papers on the problem of how many limit-cycles can exist for the generalized, mixed Rayleigh-Liénard equation. The paper of Garcia-Margallo and Bejarano[4] use the method of harmonic balance.[8] They find that Eq. (1.4) can have zero, one, or two limit-cycles. The particular number of limit-cycles and their stability is determined by the values and signs of the parameters $(A_3, B_3, D_3, E_{31}, E_{32}, F_3)$. In the second paper, Lynch[5] uses a rather complicated mixture of analytic techniques and an algorithm implemented on a computer to estimate the number of small amplitude limit-cycles for Eq. (1.4). He finds that at most three such limit-cycles can exist.

1.3 Outline of Thesis

Chapter Two gives a brief discussion of the KBM averaging method for calculating analytic approximations to nonlinear, ordinary differential equations having the form

$$\ddot{x} + x = \epsilon F(x, \dot{x}), \tag{1.5}$$

where F is, in general, a rational function of its arguments, and ϵ is a small positive parameter, i.e.,

$$0 < \epsilon \ll 1. \tag{1.6}$$

The procedure for determination of the limit-cycles' amplitudes and frequency is given.[9]

In Chapter Three, the generalized, mixed Rayleigh-Liénard equation is normalized to the form given in Eq. (1.5). We then calculate certain functions required for the determination of the number of limit-cycles.

Chapter Four gives a detailed discussion of the number of limit-cycles obtained from the first and second approximations of the KBM method. The existence of spurious limit-cycles is noted. The chapter ends with a summary of major results and a suggestion as to how this research can be extended.

CHAPTER TWO

THE KBM METHOD

2.1 Derivation of the First Approximation

The KBM first approximation to the solution of the differential equation

$$\ddot{x} + x = \epsilon F(x, \dot{x}), \qquad 0 < \epsilon \ll 1, \tag{2.1}$$

is obtained by assuming that $x(t,\epsilon)$ and its derivative $\dot{x}(t,\epsilon)$ can be written as

$$x = a(t,\epsilon)\cos[t + \phi(t,\epsilon)], \tag{2.2}$$

$$\dot{x} = -a(t,\epsilon)\sin[t + \phi(t,\epsilon)]. \tag{2.3}$$

The amplitude, $a(t,\epsilon)$, and phase, $\phi(t,\epsilon)$, are to be determined.

Let

$$\psi(t,\epsilon) = t + \phi(t,\epsilon). \tag{2.4}$$

Differentiating Eq. (2.2) and using the result of Eq. (2.3) gives one relation that $\dot{a}$ and $\dot{\phi}$ must satisfy:

$$\dot{a}\cos\psi - a\dot{\phi}\sin\psi = 0. \tag{2.5}$$

Differentiating Eq. (2.3) and substituting it and Eq. (2.3) into Eq. (2.1) gives upon simplification a second relation between $\dot{a}$ and $\dot{\phi}$:

$$\dot{a}\sin\psi + a\dot{\phi}\cos\psi = -\epsilon F(a\cos\psi, -a\sin\psi). \tag{2.6}$$

Equations (2.5) and (2.6) are two linear equations for $\dot{a}$ and $\dot{\phi}$. Solving them gives the following expressions for $\dot{a}$ and $\dot{\phi}$:

$$\frac{da}{dt} = -\epsilon F(a\cos\psi, -a\sin\psi)\sin\psi, \tag{2.7}$$

$$\frac{d\phi}{dt} = -\left(\frac{\epsilon}{a}\right) F(a\cos\psi, -a\sin\psi)\cos\psi. \tag{2.8}$$

These are exact equations for the functions $a(t,\epsilon)$ and $\phi(t,\epsilon)$. However, these are coupled, nonlinear, ordinary differential equations and, in general, are quite complicated.

To proceed, note that the right-sides of Eqs. (2.7) and (2.8) are periodic functions of ψ with a period of 2π. This is a consequence of ψ appearing only in the functional forms $\cos\psi$ and $\sin\psi$. Thus, the following Fourier series representations hold[9]:

$$F\sin\psi = K_0(a) + \sum_{n=1}^{\infty} \left[K_n(a)\cos(n\psi) + L_n(a)\sin(n\psi)\right], \tag{2.9}$$

$$F\cos\psi = P_0(a) + \sum_{n=1}^{\infty} \left[P_n(a)\cos(n\psi) + Q_n(a)\sin(n\psi)\right]. \tag{2.10}$$

The first approximation of the KBM method consists of neglecting all the terms on the right-side of Eqs. (2.9) and (2.10) except for the first. This means that, in this approximation, Eqs. (2.7) and (2.8) become

$$\frac{da}{dt} = -\epsilon K_0(a) = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\sin\psi\, d\psi, \tag{2.11}$$

$$\frac{d\phi}{dt} = -\left(\frac{\epsilon}{a}\right) P_0(a) = -\left(\frac{\epsilon}{2\pi a}\right) \int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\cos\psi\, d\psi. \tag{2.12}$$

In the research literature, the standard notation for Eqs. (2.11) and (2.12) is

$$\frac{da}{dt} = \epsilon A_1(a), \tag{2.13}$$

$$\frac{d\phi}{dt} = \epsilon B_1(a), \tag{2.14}$$

where

$$A_1(a) = -\left(\frac{1}{2\pi}\right) \int_0^{2\pi} F \sin\psi\, d\psi, \tag{2.15}$$

$$B_1(a) = -\left(\frac{1}{2\pi a}\right) \int_0^{2\pi} F \cos\psi\, d\psi. \tag{2.16}$$

2.2 Second Approximation

The second approximation for the amplitude and phase is given by the expressions[7,8]

$$\frac{da}{dt} = \epsilon A_1(a) + \epsilon^2 A_2(a), \tag{2.17}$$

$$\frac{d\phi}{dt} = \epsilon B_1(a) + \epsilon^2 B_2(a), \tag{2.18}$$

where the solution $x(t, \epsilon)$ is

$$x(t, \epsilon) = a\cos\psi + \epsilon u_1(a, \psi). \tag{2.19}$$

The functions $A_1(a)$ and $B_1(a)$ are as given in Eqs. (2.15) and (2.16). The functions $u_1(a,\psi)$, $A_2(a)$ and $B_2(a)$ are calculated using the following procedure[7,8]:

(1) Expand $F(a\cos\psi, -a\sin\psi)$ into its Fourier series:

$$F(a\cos\psi, -a\sin\psi) = g_0(a) + \sum_{n=1}^{\infty} \left[g_n(a)\cos(n\psi) + h_n(a)\sin(n\psi)\right]. \tag{2.20}$$

Since $F(a\cos\psi, -a\sin\psi)$ is a known function, the Fourier coefficients $g_n(a)$ and $h_n(a)$ can always be determined.

(2) The function $u_1(a,\psi)$ is given by the expression

$$u_1(a,\psi) = g_0(a) + \sum_{n=2}^{\infty} \left[\frac{g_n(a)\cos(n\psi) + h_n(a)\sin(n\psi)}{1-n^2}\right]. \tag{2.21}$$

(3) Consider the function $F_1(a,\psi)$ defined to be the following expression:

$$\begin{aligned} F_1(a,\psi) = {} & u_1 F_x + \left[A_1\cos\psi - aB_1\sin\psi + \frac{\partial u_1}{\partial\psi}\right] F_{\dot{x}} \\ & + \left(aB_1^2 - A_1\frac{dA_1}{da}\right)\cos\psi + \left(2A_1B_1 + A_1\frac{dB_1}{da}a\right)\sin\psi \\ & - 2A_1\frac{\partial^2 u_1}{\partial a\partial\psi} - 2B_1\frac{\partial^2 u_1}{\partial\psi^2}, \end{aligned} \tag{2.22}$$

where the following notation is used

$$F_x \equiv \frac{\partial F(x,\dot{x})}{\partial x}, \qquad F_{\dot{x}} \equiv \frac{\partial F(x,\dot{x})}{\partial \dot{x}}. \tag{2.23}$$

Note that $F_1(a,\psi)$ is known since A_1, B_1 and u_1 are given respectively by Eqs. (2.15), (2.16) and (2.21). The function $F_1(a,\psi)$ has the Fourier expansion

$$F_1(a,\psi) = g_0^{(1)}(a) + \sum_{n=1}^{\infty} \left[g_n^{(1)}(a)\cos(n\psi) + h_n^{(1)}(a)\sin(n\psi)\right]. \tag{2.24}$$

(4) The functions $A_2(a)$ and $B_2(a)$ are given by

$$A_2(a) = -\left[\frac{h_1^{(1)}(a)}{2}\right], \tag{2.25}$$

$$B_2(a) = -\left[\frac{g_1^{(1)}(a)}{2a}\right]. \tag{2.26}$$

2.3 Limit-Cycles

The existence of limit-cycles is determined by setting the derivative of $a(t,\epsilon)$ equal to zero. For example, in the first approximation, this means that the following equation must be solved for the stationary amplitudes $\bar{a}$:

$$A_1(\bar{a}) = 0. \tag{2.27}$$

Denote the solutions to this equation by

$$\{\bar{a}^{(i)}\}, \qquad i = 1, \ldots, I; \tag{2.28}$$

where I is a non-negative integer. The analytic approximations to the limit-cycles are[6,7,9]

$$x^{(i)}(t,\epsilon) = \bar{a}^{(i)} \cos[\omega^{(i)} t + \phi_0], \tag{2.29}$$

where $i = 1, \ldots, I$; $\phi_0^{(i)}$ is a constant, and $\omega^{(i)}$ is

$$\omega^{(i)} = 1 + \epsilon B_1[\bar{a}^{(i)}]. \tag{2.30}$$

CHAPTER THREE

THE RAYLEIGH-LIENARD EQUATION

3.1 Preliminaries

The generalized, mixed Rayleigh-Liénard equation is

$$\ddot{x} + A_3 x + B_3 x^3 = (D_3 + E_{31} x^2 + E_{32} \dot{x}^2 + F_3 x^4)\dot{x}. \tag{3.1}$$

Without loss of generality, the coefficient A_3 can be set equal to one. The application of the KBM method requires that all the other coefficients be of order ϵ, where

$$0 < \epsilon \ll 1. \tag{3.2}$$

Making the requirements

$$B_3 = \epsilon b_3, \quad D_3 = \epsilon d_3, \quad E_{31} = \epsilon e_{31}, \quad E_{32} = \epsilon e_{32}, \quad F_3 = \epsilon f_3, \tag{3.3}$$

Eq. (3.1) becomes

$$\ddot{x} + x = \epsilon \left\{ -b_3 x^3 + [d_3 + e_{31} x^2 + e_{32} \dot{x}^2 + f_3 x^4]\dot{x} \right\}. \tag{3.4}$$

This equation is now in the form given by Eq. (2.1) with

$$F(x, \dot{x}) = -b_3 x^3 + [d_3 + e_{31} x^2 + e_{32} \dot{x}^2 + f_3 x^4]\dot{x}. \tag{3.5}$$

3.2 Calculation of $A_1(a)$ and $B_1(a)$

Substitution of Eq. (3.5) into Eqs. (2.15) and (2.16) and evaluating the elementary trigonometric integrals, gives the following expressions for $A_1(a)$ and $B_1(a)$:

$$A_1(a) = \left(\frac{a}{2}\right)\left[d_3 + \left(\frac{e_{31} + 3e_{32}}{4}\right)a^2 + \left(\frac{f_3}{8}\right)a^4\right], \tag{3.6}$$

$$B_1(a) = \frac{3b_3a^2}{8}. \tag{3.7}$$

Therefore, to first approximation, the amplitude and phase equations are

$$\frac{da}{dt} = \epsilon A_1(a) = \epsilon\left(\frac{a}{2}\right)\left[d_3 + \left(\frac{e_{31} + 3e_{32}}{4}\right)a^2 + \left(\frac{f_3}{8}\right)a^4\right], \tag{3.8}$$

$$\frac{d\phi}{dt} = \epsilon B_1(a) = \epsilon\left(\frac{3b_3a^2}{8}\right). \tag{3.9}$$

Observe that $A_1(a)$ is a fifth-order polynomial in a, while $B_1(a)$ is a second-order polynomial in a.

3.3 $A_2(a)$ and $B_2(a)$

The functions $A_2(a)$ and $B_2(a)$ have been calculated using $A_1(a)$ and $B_1(a)$ given by Eqs. (3.6) and (3.7), and using the Fourier expansion of $F_1(a, \psi)$. The expressions obtained are long and complicated. For our purposes, we need only to know that $A_2(a)$ is a polynomial function of order seven, while $B_2(a)$ is a polynomial of order eight. They can be represented by polynomials having the following structures:

$$A_2(a) = a[\alpha_1 + \alpha_2 a^2 + \alpha_3 a^4 + \alpha_4 a^6], \tag{3.10}$$

$$B_2(a) = \beta_1 + \beta_2 a^2 + \beta_3 a^4 + \beta_4 a^6 + \beta_5 a^8. \tag{3.11}$$

The various coefficients

$$\{\alpha_i\}, \quad i = 1,2,3,4; \qquad \{\beta_j\}, \quad j = 1,2,3,4,5, \tag{3.12}$$

depend on $(b_3, d_3, e_{31}, e_{32}, f_3)$.

CHAPTER FOUR

DISCUSSION

4.1 Number of Limit-Cycles: First Approximation

For the KBM first approximation, the limit-cycles are determined by the condition[6,7,9]

$$A_1(\bar{a}) = \left(\frac{\bar{a}}{2}\right)\left[d_3 + \left(\frac{e_{31} + 3e_{32}}{4}\right)\bar{a}^2 + \left(\frac{f_3}{8}\right)\bar{a}^4\right]. \tag{4.1}$$

The solution $\bar{a}_{(1)} = 0$ is a limit-point[6,7,9] and corresponds to a limit-cycle of zero radius. It also corresponds to a fixed-point of the differential equation, i.e., $(x, \dot{x}) = (0, 0)$.

Since $A_1(\bar{a})/\bar{a}$ is quadratic in $\bar{a}^2$, if real $\bar{a}_{(2)}$ is a solution, then $-\bar{a}_{(2)}$ is also a solution. However, $\bar{a}_{(2)}$ and $-\bar{a}_{(2)}$ correspond to the same limit-cycle since they differ by at most a constant change in the phase function.[7,9] Putting all these facts together, it can be concluded that Eq. (4.1) has at most three real, non-negative solutions. This implies that the generalized, mixed Rayleigh-Liénard equation can have at most two limit-cycles; the limit-point, $\bar{a}_{(1)} = 0$, is always present.

4.2 Number of Limit-Cycles: Second Approximation

The number of limit-cycles obtained using the KBM second approximation is determined by the condition

$$A_1(\bar{a}) + \epsilon A_2(\bar{a}) = 0. \tag{4.2}$$

In sections 3.2 and 3.3, it was shown that A_1 and A_2 are polynomials of respective orders five and seven. With a change of notation, Eq. (4.2) can be rewritten to the form

$$P_5(\bar{a}) + \epsilon P_7(\bar{a}) = 0, \tag{4.3}$$

where $P_n(\bar{a})$ is an n-th order polynomial in $\bar{a}$. Since $\bar{a}$ is a factor of both $P_5(\bar{a})$ and $P_7(\bar{a})$, Eq. (4.3) can be written as

$$\bar{a}[P_2(\bar{a}^2) + \epsilon P_3(\bar{a}^2)] = 0, \tag{4.4}$$

where as indicated, P_2 and P_3 are functions only of $\bar{a}^2$.

Inspection of Eq. (4.4) leads to the conclusion that three limit-cycles might exist (in addition to the limit-point at $\bar{a} = 0$). This contradicts what was obtained using the first order KBM method. The number of actual limit-cycles should not depend on the order of the method used. How can this paradox be resolved[2]? The next section provides the answer.

4.3 Spurious Limit-Cycles

The additional possible limit-cycle that arises in the second approximation is called a "spurious limit-cycle."[10] It corresponds to a root $\bar{a}$ that has a singular dependence on the small parameter ϵ, i.e.,

$$\bar{a}(\epsilon) = \frac{\text{constant}}{\epsilon^{\gamma}}, \tag{4.5}$$

where the constant γ satisfies the restriction

$$0 < \gamma \leq 1. \tag{4.6}$$

Note that as $\epsilon \to 0$, $\bar{a}(\epsilon) \to \infty$. In general, if the amplitude expression $A(a)$, as it occurs in

$$\frac{da}{dt} = A(a, \epsilon), \tag{4.7}$$

could be calculated to all orders of ϵ, then the roots of

$$A(\bar{a}, \epsilon) = 0, \qquad \{\bar{a}^{(i)}(\epsilon)\}, \qquad i = 1, 2, \ldots, I; \tag{4.8}$$

would have the property

$$\bar{a}^{(i)}(\epsilon) = \bar{a}^{(i)}(0) + O(\epsilon). \tag{4.9}$$

This means that in the full, exact calculation, the singular roots must cancel.[10]

4.4 Summary and Extension of Research

The results of sections 4.1, 4.2 and 4.3 lead to the following conclusion: The generalized, mixed Rayleigh-Liénard equation can have a maximum of two limit-cycles. The exact number existing will depend on the particular values and signs of the parameters that occur in the differential equation. Our results are consistent with the recent work of Garcia-Margallo and Bejarano[4], and Lynch[5]. A brief exposition of our calculations has been accepted for publication.[11]

In general, perturbation calculations are done to finite order in ϵ. It would be of value to further investigate the phenomena of spurious limit-cycles using analytic approximation techniques other than the KBM method.

REFERENCES

1. C. R. Wylie and L. C. Barrett, *Advanced Engineering Mathematics* (McGraw-Hill, New York, 1995, 6th edition); Chapter 6.

2. A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Qualitative Theory of Second-Order Dynamic Systems* (Wiley, New York, 1973).

3. Lord Rayleigh, *The Theory of Sound* (Dover, New York, 1945 reprint).

4. J. Garcia-Margallo and J. D. Bejarano, *Journal of Sound and Vibration* **156**, 283–301 (1992). "The Limit-Cycles of the Generalized Rayleigh-Liénard Oscillator."

5. S. Lynch, *Journal of Sound and Vibrations* **178**, 615–620 (1994). "Small Amplitude Limit-Cycles of the Generalized Mixed Rayleigh-Liénard Oscillator."

6. N. Krylov and N. Bogoliubov, *Introduction to Nonlinear Mechanics* (Princeton University Press; Princeton, NJ; 1943).

7. N. N. Bogoliubov and J. A. Mitropolsky, *Asymptotical Methods in the Theory of Nonlinear Oscillators* (Hindustan Publishing Company; Delhi, India; 1963).

8. R. E. Mickens, *Journal of Sound and Vibration* **94**, 456–460 (1984). "Comments on the Method of Harmonic Balance."

9. R. E. Mickens, *Nonlinear Oscillations* (Cambridge University Press, New York, 1981); Chapter 3.

10. J. Burnette and R. E. Mickens, "Spurious Limit-Cycles in Higher Order Averaging Methods," unpublished.

11. J. Burnette and R. E. Mickens, *Journal of Sound and Vibration* (accepted for publication), "The Number of Limit-Cycles for the Generalized Mixed Rayleigh-Liénard Oscillator."